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Regular Embeddings of Canonical Double Coverings of Graphs

ROMAN NEDELA

Department of Mathematics, Faculty of Science, Matej Bel University, 975 49 Banská Bystrica, Slovakia

AND

MARTIN ŠKOVIERA

Department of Computer Science, Faculty of Mathematics and Physics,

This paper addresses the question of determining, for a given graph G, all regular maps having G as their underlying graph, i.e., all embeddings of G in closed surfaces exhibiting the highest possible symmetry. We show that if G satisfies certain natural conditions, then all orientable regular embeddings of its canonical double covering, isomorphic to the tensor product $G \otimes K_2$, can be described in terms of regular embeddings of G. This allows us to "lift" the classification of regular embeddings of a given graph to a similar classification for its canonical double covering and to establish various properties of the "derived" maps by employing those of the "base" maps. We apply these results to determining all orientable regular embeddings of the tensor products $K_n \otimes K_2$ (known as the cocktail-party graphs) and of the n-dipoles D_n, the graphs consisting of two vertices and n parallel edges joining them. In the first case we show, in particular, that regular embeddings of $K_n \otimes K_2$ exist only if n is a prime power p^l, and there are $2\phi(n-1)$ or $\phi(n-1)$ isomorphism classes of such maps (where ϕ is Euler's function) according to whether l is even or odd. For l even an interesting new infinite family of regular maps is discovered. In the second case, orientable regular embeddings of D_n exist for each positive integer n, and their number is a power of 2 depending on the decomposition of n into primes.

1. INTRODUCTION

In this paper we examine the problem of determining and classifying all regular maps on closed surfaces which have the same underlying graph. By a map we mean a pair (G, S) where G is a finite graph embedded in a closed orientable surface S in such a way that every connected component

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of $S - G$ (a face) is simply-connected; a map is regular if the group of all orientation-preserving map automorphisms acts transitively (and hence regularly) on the oriented edges of G.

All regular maps having the same underlying graph have been determined only for a very restricted class of graphs, most notably for complete graphs K_n (James and Jones [9]) and for bouquets of circles (Škoviera and Širáň [17], in the dual form see also Garbe [5, Satz 6.2] and Vince [18]). Some other important graphs, like the complete bipartite graphs $K_{n,n}$, the complete tripartite graphs $K_{n,n,n}$ or the n-cubes Q_n, are known to have regular embeddings for each n (see [19]), but the description of *all* of them does not seem to be at hand.

It appears that this problem might be too difficult to be dealt with in general, so one could try to consider it for another (preferably infinite) family of graphs. Instead of doing this, we approach the problem by asking if one can "lift" a classification of all regular embeddings of a given graph G to a classification of all regular embeddings of some related graph G', in particular, a covering of G. The simplest instance of this situation occurs when G' is taken to be the canonical double covering $\tilde{G}$. The canonical double covering of a graph G is constructed by taking two copies v_0 and v_1 of every vertex v of G and joining, for every edge uw of G, u_0 to w_1 and u_1 to w_0; thus $\tilde{G}$ is isomorphic to the tensor product $G \otimes K_2$. The advantage of this construction is that it applies uniformly to every graph and has many nice properties. In this context, two of them are worth of mention.

(1) Every automorphism of G lifts to an automorphism of $\tilde{G}$; it follows that the automorphism group of $\tilde{G}$, $\mathrm{Aut}(\tilde{G})$, contains a subgroup isomorphic to $\mathrm{Aut}(G) \times \mathbb{Z}_2$.

(2) Every regular embedding of G lifts to a regular embedding of $\tilde{G}$; if $\tilde{M}$ is the lifted map and M is the base, then it is easy to see that $\mathrm{Aut}(\tilde{M}) \cong \mathrm{Aut}(M) \times \mathbb{Z}_2$.

These facts suggest that for $G' = \tilde{G}$ our programme of determining all regular embeddings of G' could be realizable.

Observe, however, that the canonical double covering graph $\tilde{G}$ may have additional regular embeddings which, of course, do not cover any orientable regular embedding of G—as is the case with the 3-dimensional cube $Q_3 = K_4 \otimes K_2$. Here the base graph K_4 underlies only one orientable regular map—the tetrahedron. This map lifts to a hexagonal embedding of Q_3 in the torus and not, as one might expect, to the familiar quadrilateral embedding in the 2-sphere—the hexahedron.

Nevertheless, the latter regular map can be obtained from the tetrahedron by another construction which applies in general as long as the base map is reflexible. Roughly speaking, one has to mimic the base map only in one of the copies of the vertex-set, whereas the other copy must employ the *mirror image* of the base map instead. To put it differently, if R_v is the local rotation at some vertex v of the base map M, i.e., the cyclic permutation of the incident edges induced by M, then the local rotations at v_0 and v_1 correspond to R_v and R_v^{-1}, respectively.

This idea can be further generalized: we may replace R_v^{-1} by appropriate powers of R_v and still obtain regular embeddings of the canonical double covering.

The aim of the present paper is to show that the resulting regular embeddings of the canonical double covering graph $\tilde{G}$ constitute the complete set of regular embeddings in the important special case when $\mathrm{Aut}(\tilde{G}) \cong \mathrm{Aut}(G) \times \mathbb{Z}_2$, that is to say, to realize our programme in this case. Graphs G with $\mathrm{Aut}(\tilde{G}) \cong \mathrm{Aut}(G) \times \mathbb{Z}_2$ have recently been investigated by Marušič *et al.* in [12, 13] and termed *stable*. As the results of [13] suggest, not many vertex-transitive unstable graphs seem to exist, so the assumption of stability is not very restrictive. This indicates a potentially wide applicability of our results. Unfortunately, they cannot be used repeatedly because $\tilde{G}$ is always bipartite, and therefore its canonical double covering is disconnected, consisting of two copies of $\tilde{G}$.

Our paper is organized as follows. The next three sections have a preliminary character. In Section 2 we briefly review the necessary graph-theoretic and map-theoretic terminology, slightly modified for our purposes. In Section 3 we introduce a new concept of an exponent of a graph or of a map, which will be extremely useful in Section 5 where the main results of this paper—the classification theorems—are proved. Before doing that, we look in Section 4 at the automorphism group of the canonical double covering graph and discuss the notion of stability, which will then be used throughout the paper. Section 6 and Section 7 are devoted to the symmetry properties of the derived regular maps, in particular to their automorphism groups and exponent groups. The final two sections illustrate how the general classification theorems apply in two special cases: the cocktail-party graphs $K_n \otimes K_2$ (Section 8), and the n-dipoles, graphs consisting of two vertices joined by n parallel edges (Section 9).

2. Graphs and Maps

Graphs considered in this paper are finite and have no isolated vertices. Edges of our graphs are one of three types: *links*, *loops*, and *semiedges*. Multiple adjacencies are permitted. A link is incident with two different vertices while a loop or a semiedge is incident with a single vertex. A link or a loop gives rise to two oppositely directed arcs that are *reverse* to each other. A semiedge incident with a vertex u gives rise to a single arc initiating at u and is reverse to itself. We use $L(x)$ to denote the reverse to an arc x. Thus, if $D(G)$ is the set of all arcs of a graph G, then L is an involutory permutation of $D(G)$, i.e., $L^2 = \mathrm{id}$.

For the purpose of this paper it is useful to view a graph G as a quadruple $G = (D, V, I, L)$, where $D = D(G)$ and $V = V(G)$ are disjoint non-empty finite sets, $I: D \to V$ is a surjective mapping, and L is an involution on D. The elements of D and V are *arcs* and *vertices*, respectively, I is the incidence function assigning to every arc its *initial vertex*, and L is the *arc-reversing involation*; the orbits of the group $\langle L \rangle$ are edges of G. Topologically speaking, our graphs are nothing but 1-dimensional cell complexes.

The usual graph-theoretical concepts such as walks, cycles, connectedness, biparticity, etc. transfer to our graphs in the obvious way. In particular, the *valency* of a vertex v is the number of arcs having v as their initial vertex.

Let $G = (D, V, I, L)$ and $G' = (D', V', I', L')$ be two graphs. A *homomorphism* $\alpha: G \to G'$ is a mapping $\alpha: D \cup V \to D' \cup V'$ with $\alpha(D) \subseteq D'$ and $\alpha(V) \subseteq V'$ such that $\alpha I = I'\alpha$ and $\alpha L = L'\alpha$. As usual, an *automorphism* of G is an isomorphism $G \to G$. We use $\mathrm{Aut}(G)$ to denote the automorphism group of G. Of course, if G is simple, then our notion of an automorphism corresponds to the usual one (i.e., to a vertex-automorphism). In general, however, it may happen that a non-trivial automorphism induces the trivial vertex-automorphism. Consequently, our automorphism group may not coincide with the usual vertex-automorphism group. For instance, if the *n-semistar* Ss_n is the graph consisting of a single vertex and n semiedges incident with it, then $\mathrm{Aut}(Ss_n) \cong S_n$, the symmetric group on n elements, while its vertex-automorphism group is trivial.

A map is a connected (topological) graph cellularly embedded in some closed surface, i.e., every face is homeomorphic to an open disc. In this paper we only consider maps on orientable surfaces. As with graphs, it is often convenient to replace a topological embedding by its combinatorial description. Therefore, by a *map* we henceforth mean a pair $M = (G, P)$, where G is a connected graph and P is a permutation of $D(G)$, called *rotation*, cyclically permuting arcs with the same initial vertex, i.e., $IP(x) = I(x)$ for every $x \in D(G)$. The cyclic permutation P_v permuting the arcs emanating from a vertex v is the *local rotation* at v. Our definition of a map thus essentially coincides with the definition of a finite algebraic map given in Jones and Singerman [10], where the reader can also find the details concerning the relationship between topological maps and combinatorial (or algebraic) maps.

It is easy to see that to describe a map M one only needs to specify the arc-set $D = D(M)$, the rotation P and the arc-reversing involution L, because the vertices of G bijectively correspond to the cycles of P, and the incidence function I can be replaced by the mapping which to every arc assigns the cycle of P it belongs to. The fact that G is connected is reflected by the transitive action of the permutation group $\langle P, L\rangle$, which we call the *monodromy group* of M and denote by $\mathrm{Mon}(M)$. (In [15] the same group is denoted by $G(M)$.) This usually allows us to omit vertices from consideration whenever we use such a description of a map.

Let $M=(G, P)$ and $M'=(G', P')$ be maps. We define a map *isomorphism* α: $M \to M'$ to be an isomorphism $G \to G'$ such that $\alpha P = P'\alpha$. In the light of the preceding remark we can identify an isomorphism $(G, P) \to (G', P')$ with a bijection α on the arc-sets which satisfies the condition $\alpha P = P'\alpha$ and $\alpha L = L'\alpha$. An *automorphism* α of M is then a self-isomorphism $M \to M$, i.e., $\alpha P = P\alpha$ and $\alpha L = L\alpha$.

Because the monodromy group acts transitively on arcs, for any two arcs x and y of a map $M=(G, P)$ there is at most one map automorphism which takes x to y. Thus, for the automorphism group $\mathrm{Aut}(M)$ of M we always have $|\mathrm{Aut}(M)| \leqslant |D(G)|$. A map M is called *regular* if $|\mathrm{Aut}(M)| = |D(G)|$. In other words, a map is regular if and only if $\mathrm{Aut}(M)$ acts transitively (or, equivalently, regularly) on $D(G)$. Another description of the regularity of a map is based on its monodromy group: M is regular if and only if $\mathrm{Mon}(M)$ acts regularly on $D(G)$.

The following standard construction of a regular map with given monodromy group is often very useful. Clearly, the monodromy group of every (regular) map is a finite group generated by two generators, one of which is an involution. If a finite group $H = \langle r, l;\, r^m = l^2 = (rl)^n = 1, \ldots\rangle$ is given, we can construct a regular map $A(H; r, l)$ in the following way: we take elements of H as arcs and define the arc-reversing involution L and the rotation R by setting $L(x) = lx$ and $R(x) = rx$, $x \in H$, respectively. Denoting, for a given element $h \in H$, by ξ_h the permutation $x \mapsto xh$, $x \in H$, we see that $R\xi_h = \xi_h R$ and $L\xi_h = \xi_h L$, i.e., ξ_h is an automorphism of $A(H; r, l)$. It follows that $|\mathrm{Aut}(A(H; r, l))| \geqslant |H| = |D(A(H; r, l))|$ implying that the map is regular with $\mathrm{Aut}(A(H; r, l)) = \{\xi_h;\ h \in H\}$. Moreover, $\mathrm{Mon}(A(H; r, l))$ is equal to H acting on itself by left translation. In particular, $\mathrm{Aut}(A(H; r, l)) \cong \mathrm{Mon}(A(H; r, l))$.

On the other hand, if M is an arbitrary regular map with rotation R and arc-reversing involution L, then we establish an isomorphism $M \to A(\mathrm{Mon}(M); R, L)$ as follows: we fix an arc z of M and label every arc x by the unique element $w \in \mathrm{Mon}(M)$ for which $w(z) = x$, thereby obtaining a labelling $\lambda(x) = w$. Clearly, $\lambda(R(x)) = R.\lambda(x)$ and $\lambda(L(x)) = L.\lambda(x))$ which proves that λ is the required isomorphism $M \to A(\mathrm{Mon}(M); R, L)$. As a consequence we obtain that $\mathrm{Mon}(M) \cong \mathrm{Aut}(M)$ for every regular map M.

Besides map automorphisms, reflections of maps are also often considered. A map $M=(G, P)$ is called *reflexible* if it is isomorphic to its *mirror image* (G, P^{-1}); an isomorphism α: $(G, P) \to (G, P^{-1})$ is called a *reflection* of M. (Note that in a different terminology, used e.g. by Wilson [20, 21], our regular maps are called "rotary" while the term "regular map" is used to what we call "regular and reflexible map".)

Instead of P^{-1} one could well take other powers of P and ask whether they produce maps isomorphic to (G, P). This idea, leading to a natural generalization of reflexibility, is very useful and will be briefly discussed in the next section.

3. Exponents of Graphs and Maps

Let $M = (G, P)$ be a map, where G is a connected graph and P is a rotation of G. An integer e will be said to be an *exponent* of G if P^e is again a rotation of G. An exponent e of G will be said to be an *exponent* of M if the map $M' = (G, P^e)$ is isomorphic to M. Recall that the maps (G, P^e) and (G, P) are isomorphic if and only if there exists an automorphism ψ of G such that $\psi P = P^e \psi$. The automorphism ψ of G will be said to be *associated* with the exponent e of M.

We now review some basic properties of exponents which will be used in Section 5 and later. Other results concerning exponents will be given in Section 8 and Section 9. In most cases we omit proofs for they would unduly lengthen the paper. For details the reader is referred to [16].

Observe that P^e is a rotation of G if and only if, for each vertex v of G, the local rotation $(P_v)^e$ consists of a single cycle, i.e., if $\gcd(n, e) = 1$. Thus, denoting by m the least common multiple of the valencies of vertices of G, we see that an integer e is an exponent of G if and only if it is relatively prime to m. The residue classes modulo m of exponents of G therefore form a multiplicative group which coincides with the group $\mathbb{Z}_m^*$ of invertible elements of the ring $\mathbb{Z}_m$. We denote this group by $\mathrm{Ex}(G)$ and call it the *exponent group* of the graph G. The order of $\mathrm{Ex}(G)$ is obviously $\phi(m)$, where ϕ is Euler's integer function.

As far as exponents of maps are concerned, it is clear that 1 is an exponent of every map M, and -1 is an exponent of M if and only if M is reflexible. It is also easily seen that if ψ and ξ are automorphisms of the graph G associated with the exponents e and f of M, respectively, then $\xi\psi$ (and also $\psi\xi$) is associated with ef. It follows that the product of two exponents of M is again an exponent of M. Consequently, the residue classes of exponents of $M = (G, P)$ form a subgroup of $\mathrm{Ex}(G)$ which we denote by $\mathrm{Ex}(M)$ and call the *exponent group* of M. In a sense, this group reflects "external" symmetries of a map (see [16]). Examples of exponent groups can be found in Section 8 and Section 9.

The *order* of an exponent e of a graph G is the order of its residue class $[e] \in \mathrm{Ex}(G)$. An exponent of order $\leqslant 2$ will be called *involutory*. For instance, if M is a reflexible map, then -1 is an involutory exponent of M. Since $\mathrm{Ex}(G)$ and $\mathrm{Ex}(M)$ are Abelian groups, involutory exponents of M form a subgroup of $\mathrm{Ex}(M)$ denoted by $\mathrm{Ex}_2(M)$ and called the *involutory exponent group* of M. This group will be very useful in Section 5.

The following somewhat surprising result will also be needed in Section 5. It explains why exponents are so important for classifying regular embeddings of a given graph. Its proof can be found in [16, Section 5].

THEOREM 3.1. *Let $M=(G, P)$ and $M'=(G, P')$ be two regular embeddings of a connected graph G and let* $\mathrm{Aut}(M)=\mathrm{Aut}(M')$. *Then there exists an exponent e of G such that $P'=P^e$.*

To conclude this section, we should mention that the map (G, P^e) can be obtained by applying Wilson's operator H_e to the map (G, P). These operators were introduced by Wilson in [20] and were later interpreted algebraically by Jones and Thornton in [11].

4. Automorphisms of Canonical Double Coverings

The *canonical double covering* $\tilde{G}$ of a graph G is the graph with arc-set $D(G)\times\mathbb{Z}_2$, vertex-set $V(G)\times\mathbb{Z}_2$, incidence function

$$\tilde{I}(x, i)=(I(x), i), \qquad i\in\mathbb{Z}_2 \tag{4.1}$$

and arc-reversing involution

$$\tilde{L}(x, i)=(L(x), i+1), \qquad i\in\mathbb{Z}_2 \tag{4.2}$$

where $x\in D(G)$. It is easy to see that the mapping $\pi\colon \tilde{G}\to G$ which erases the second coordinate is a two-fold covering projection. Moreover, a short reflection shows that $\tilde{G}$ is connected if and only if G is not bipartite.

In this section we investigate the automorphism group $\mathrm{Aut}(\tilde{G})$ of the canonical double covering $\tilde{G}$ of G, in order to prepare for the proof of our main results about regular embeddings of these graphs.

We often think of the vertex $(v, i)\in V(\tilde{G})$ as being coloured by the element $i\in\mathbb{Z}_2$. It follows from (4.1) and (4.2) that the end-vertices of each edge of $\tilde{G}$ have distinct colours; hence $\tilde{G}$ is bipartite. In fact, $\tilde{G}$ is isomorphic to the tensor product $G\otimes K_2$. Equivalently, $\tilde{G}$ can be obtained by assigning to every arc of G the voltage $1\in\mathbb{Z}_2$ and applying the derived graph construction [6].

In order to make our notation more practical, we shall usually write z_i for (z, i), z being a vertex or an arc of G. Accordingly, we simplify $V(G)\times\{i\}$ to V_i.

Let H be a bipartite graph with bipartition $V(H) = V_0 \cup V_1$. An automorphism $\psi \in \mathrm{Aut}(H)$ will be called *colour-preserving* if ψ maps V_i onto V_i for $i \in \mathbb{Z}_2$, and *colour-reversing* if it maps V_i onto V_{i+1} for $i \in \mathbb{Z}_2$. In general, $\mathrm{Aut}(H)$ may also contain "mixed" automorphisms, however, it is a straightforward matter to see that this is not the case if H is connected.

LEMMA 4.1. *Every automorphism of a connected bipartite graph is either colour-preserving or colour-reversing.*

As far as canonical double coverings are concerned, we can say a little bit more. Namely, for an arbitrary graph G, the automorphism group $\mathrm{Aut}(\tilde{G})$ contains a subgroup isomorphic to the direct product $\mathrm{Aut}(G) \times \mathbb{Z}_2$. Indeed, for any automorphism $\psi \in \mathrm{Aut}(G)$ the mapping $\tilde{\psi}\colon \tilde{G} \to \tilde{G}$, defined by

$$\tilde{\psi}(x_i) = (\psi(x))_i, \qquad x_i \in D(\tilde{G}), \tag{4.3}$$

is a colour-preserving automorphism of $\tilde{G}$, which follows from the following computations:

$$\tilde{\psi}\tilde{I}(x_i) = \tilde{\psi}(I(x)_i) = (\psi I(x))_i = (I\psi(x))_i = \tilde{I}(\psi(x)_i) = \tilde{I}\tilde{\psi}(x_i),$$
$$\tilde{\psi}\tilde{L}(x_i) = \tilde{\psi}(L(x)_{i+1}) = (\psi L(x))_{i+1} = (L\psi(x))_{i+1} = \tilde{L}(\psi(x)_i) = \tilde{L}\tilde{\psi}(x_i).$$

Moreover, the assignment $\Theta\colon \psi \mapsto \tilde{\psi}$ is obviously a monomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(\tilde{G})$. On the other hand, $\mathrm{Aut}(\tilde{G})$ contains a colour-reversing automorphism $\beta \notin \Theta\mathrm{Aut}(G)$ given by

$$\beta(x_i) = x_{i+1}, \qquad x_i \in D(\tilde{G}). \tag{4.4}$$

Observe that for any $\psi \in \mathrm{Aut}(G)$ and $x_i \in D(\tilde{G})$ it holds that

$$\tilde{\psi}\beta(x_i) = \tilde{\psi}(x_{i+1}) = \psi(x)_{i+1} = \beta(\psi(x)_i) = \beta\tilde{\psi}(x_i).$$

Hence,

$$\tilde{\psi}\beta = \beta\tilde{\psi} \tag{4.5}$$

for any $\psi \in \mathrm{Aut}(G)$. Thus $\mathrm{Aut}(\tilde{G})$ contains the subgroup $\Theta(\mathrm{Aut}\, G) \times \langle \beta \rangle \cong \mathrm{Aut}(G) \times \mathbb{Z}_2$, as claimed.

It may happen that $\mathrm{Aut}(\tilde{G})$ contains additional automorphisms, but the important special case is when it does not. Following Marušič *et al.* [12], we call a graph G *stable* if $\mathrm{Aut}(\tilde{G})$ is isomorphic to $\mathrm{Aut}(G) \times \mathbb{Z}_2$; otherwise we call it *unstable*.

Note that if G is bipartite, then $\tilde{G}$ is disconnected and hence G is unstable. Thus a stable graph is necessarily non-bipartite. Moreover, Marušič *et al.* [12] observed that each simple stable graph is vertex-determining, i.e., it does not contain two distinct vertices with identical neighbourhoods. Although the concept of stability applies to general graphs, it naturally arises in the context of vertex-transitive ones. Complete graphs K_n and semistars Ss_n provide examples of symmetrical stable graphs; on the other hand, the bouquet of n circles B_n is easily seen to be symmetrical, but unstable (see Sections 8 and 9). Marušič *et al.* [12] characterized simple stable graphs in matrix-theoretical terms. They further exhibited an infinite family of simple, regular, non-bipartite, vertex-determining unstable graphs [13]. Among these examples, however, none is vertex-transitive.

In the next section we shall often use the following technical result.

THEOREM 4.2. *Let G be a stable graph and let $\pi: \tilde{G} \to G$ be the natural projection. Then*

$$\pi\psi(x_0) = \pi\psi(x_1), \tag{4.6}$$

for every $\psi \in \mathrm{Aut}(\tilde{G})$ and every $x \in D(G)$.

Proof. Since G is stable, $\mathrm{Aut}(\tilde{G}) \cong \Theta(\mathrm{Aut}\, G) \times \langle \beta \rangle$, and consequently, $\psi\beta = \beta\psi$ for every $\psi \in \mathrm{Aut}(\tilde{G})$. Hence

$$\pi\psi(x_0) = \pi\psi\beta(x_1) = \pi\beta\psi(x_1) = \pi\psi(x_1),$$

for every $x \in D(G)$. ∎

Let G be a stable graph and let $\psi \in \mathrm{Aut}(\tilde{G})$. Define a mapping $\psi': G \to G$ by setting

$$\psi'(x) = \pi\psi(x_0), \tag{4.7}$$

for $x \in D(G)$. We show that $\psi' \in \mathrm{Aut}(G)$.

COROLLARY 4.3. *Let G be a stable graph and let ψ be an automorphism of $\tilde{G}$. Then ψ' is an automorphism of G.*

Proof. Since the covering projection $\pi: \tilde{G} \to G$ is a graph homomorphism, we have

$$\pi\tilde{I} = I\pi \qquad \text{and} \qquad \pi\tilde{L} = L\pi.$$

Now,

$$\psi' I(x) = \pi\psi(I(x)_0) = \pi\psi\tilde{I}(x_0) = \pi\tilde{I}\psi(x_0) = I\pi\psi(x_0) = I\psi'(x)$$

and

$$\psi' L(x) = \pi\psi(L(x)_0) = \pi\psi\tilde{L}(x_1) = \pi\tilde{L}\psi(x_1)$$
$$= L\pi\psi(x_1) = L\pi\psi(x_0) = L\psi'(x).$$

The claim follows. ∎

5. Classification Theorems for Regular Embeddings of Canonical Double Coverings of Graphs

In this section we establish three important results about regular embeddings of canonical double coverings of graphs: Theorem 5.1 providing a construction of regular embeddings of the canonical double covering from regular embeddings of the base graph; Theorem 5.2 showing that every regular embedding of the canonical double covering of a stable graph is one of those constructed in Theorem 5.1; and Theorem 5.3 giving conditions under which two such regular maps are isomorphic.

Construction. Let G be a connected non-bipartite graph and let $M = (G, Q)$ be a (not necessarily regular) embedding of G. For any exponent e of G we define the *derived embedding* $M_e = (\tilde{G}, \tilde{Q})$ of its canonical double covering by setting

$$\tilde{Q}(x_i) = Q^{e^i}(x)_i, \tag{5.1}$$

for $x \in D(G)$ and $i = 0, 1$.

We shall prove that if M is a regular map and e is an involutory exponent of M, then the derived map is again regular.

Theorem 5.1. *Let $M = (G, Q)$ be a regular embedding of a connected non-bipartite graph G, and let e be an involutory exponent of M. Then the derived embedding $M_e = (\tilde{G}, \tilde{Q})$ defined by* (5.1) *is regular.*

Proof. It suffices to prove that $|\mathrm{Aut}(M_e)| \geqslant 2\, |\mathrm{Aut}(M)|$. To do this, we first construct a monomorphism from $\mathrm{Aut}(M)$ into $\mathrm{Aut}(M_e)$. Recall that every automorphism $\xi \in \mathrm{Aut}(G)$ lifts to a colour-preserving automorphism $\tilde{\xi} \in \mathrm{Aut}(\tilde{G})$ defined by (4.3). We show that if $\xi \in \mathrm{Aut}(M) \subseteq \mathrm{Aut}(G)$, then $\tilde{\xi} \in \mathrm{Aut}(M_e) \subseteq \mathrm{Aut}(\tilde{G})$. Indeed, realizing that $\mathrm{Aut}(G, Q) = \mathrm{Aut}(G, Q^e)$ we obtain

$$\tilde{\xi}\tilde{Q}(x_i) = \tilde{\xi}(Q^{e^i}(x)_i) = (\xi Q^{e^i}(x))_i = (Q^{e^i}\xi(x))_i = \tilde{Q}(\xi(x)_i) = \tilde{Q}\tilde{\xi}(x_i),$$

for every arc $x \in D(G)$ and every $i \in \{0, 1\}$. Thus $\tilde{\xi}\tilde{Q} = \tilde{Q}\tilde{\xi}$, whence $\tilde{\xi} \in \mathrm{Aut}(M_e)$. Moreover, the mapping $\Theta\colon \xi \mapsto \tilde{\xi}$ is obviously a monomorphism which maps $\mathrm{Aut}(M)$ into $\mathrm{Aut}(M_e)$. Hence $|\mathrm{Aut}(M_e)| \geqslant |\mathrm{Aut}(M)|$. (In fact, $|\mathrm{Aut}(M)|$ divides $|\mathrm{Aut}(M_e)|$.)

Let θ be an automorphism of G associated with the exponent e, i.e.,

$$\theta Q = Q^e \theta \tag{5.2}$$

Clearly, the mapping $\bar{\theta}$ defined by setting

$$\bar{\theta}(x_i) = \beta\tilde{\theta}(x_i) = \theta(x)_{i+1} \tag{5.3}$$

is an automorphism of $\tilde{G}$. We show that $\bar{\theta}$ is, in fact, an automorphism of the derived map M_e. In order to establish the identity $\bar{\theta}\tilde{Q}(x_i) = \tilde{Q}\bar{\theta}(x_i)$, $x_i \in D(\tilde{G})$, we distinguish two cases according to whether $i=0$ or $i=1$. If $i=0$, using (5.1), (5.2) and (5.3) we get

$$\bar{\theta}\tilde{Q}(x_0) = \bar{\theta}((Q(x))_0) = (\theta Q(x))_1 = (Q^e\theta(x))_1 = \tilde{Q}(\theta(x)_1) = \tilde{Q}\bar{\theta}(x_0).$$

Employing the fact that e is an involutory exponent of $M=(G, Q)$, for $i=1$ we similarly obtain

$$\begin{aligned}\bar{\theta}\tilde{Q}(x_1) &= \bar{\theta}((Q^e(x))_1) = (\theta Q^e(x))_0 = (Q^{e^2}\theta(x))_0\\ &= (Q\theta(x))_0 = \tilde{Q}(\theta(x)_0) = \tilde{Q}\bar{\theta}(x_1).\end{aligned}$$

Thus $\bar{\theta} \in \mathrm{Aut}(M_e)$. Observe that $\bar{\theta}$ is a colour-reversing automorphism of $\tilde{G}$ while every $\tilde{\xi}$ is a colour-preserving. Hence $|\mathrm{Aut}(M_e)| > |\mathrm{Aut}(M)|$. Now Lagrange's subgroup theorem implies that $|\mathrm{Aut}(M_e)| \geqslant 2\,|\mathrm{Aut}(M)|$, and the theorem is proved. ∎

We now reverse Theorem 5.1. The main idea would be to "split" the rotation of a regular embedding of M of $\tilde{G}$ into two rotations of G by erasing the indices. Unfortunately, in general we do not have enough information about the resulting maps $M^{(0)}$ and $M^{(1)}$ of G. However, if G is stable, then it turns out that the colour-preserving automorphisms of M induce automorphisms of both $M^{(0)}$ and $M^{(1)}$, whereas the colour-reversing automorphism of M induce isomorphisms $M^{(0)} \leftrightarrow M^{(1)}$. This enables us to prove the following:

THEOREM 5.2. *Let G be a connected stable graph and let $M=(\tilde{G}, Q)$ be a regular embedding of $\tilde{G}$. Then there exists a regular embedding N of G and an involutory exponent e of N such that $M \cong N_e$.*

Proof. For any arc x of G and for $i \in \{0, 1\} = \mathbb{Z}_2$ set

$$Q^{(i)}(x) = \pi Q(x_i), \qquad x \in D(Q). \tag{5.4}$$

Then $Q^{(i)}$ is a rotation of G and defines a map $M^{(i)} = (G, Q^{(i)})$. In order to prove the theorem we show that $M^{(0)} = (G, Q^{(0)})$ and $M^{(1)} = (G, Q^{(1)})$ are isomorphic regular maps and there exists an integer e such that $Q^{(1)} = (Q^{(0)})^e$, where $e^2 \equiv 1$ modulo the valency of G.

First of all we prove that the maps $M^{(i)}$, $i \in \{0, 1\}$, are regular. Let $\Sigma \subseteq \mathrm{Aut}(M)$ be the subgroup consisting of all colour-preserving automorphisms of M. Obviously, the index of Σ in $\mathrm{Aut}(M)$ is 2, so $|\Sigma| = \mathrm{Aut}(M)|/2$. Since G is stable, (4.6) implies that

$$\pi\psi(x_0) = \pi\psi(x_1), \tag{5.5}$$

for every $\psi \in \Sigma$ and every $x \in D(G)$. As we have proved in Corollary 4.3, the mapping $\psi'(x) = \pi\psi(x_0) = \pi\psi(x_1)$ is an automorphism of the graph G. We now show that ψ' is, in fact, a map automorphism for both $M^{(0)}$ and $M^{(1)}$, and the assignment $\psi \mapsto \psi'$ is a monomorphism $\Sigma \to \mathrm{Aut}(M^{(i)})$, $i = 0, 1$. To see this, observe that

$$\psi'\pi = \pi\psi. \tag{5.6}$$

Furthermore, ψ is colour-preserving, so

$$\psi(x_i) = (\pi\psi(x_i))_i = (\psi'(x))_i. \tag{5.7}$$

From (5.4), (5.5) and (5.7) we obtain

$$\psi' Q^{(i)}(x) = \psi'\pi Q(x_i) = \pi\psi Q(x_i) = \pi Q\psi(x_i) = \pi Q((\psi'(x))_i = Q^{(i)}\psi'(x),$$

which implies that $\psi' \in \mathrm{Aut}(M^{(i)})$, for $i = 0, 1$. Consequently,

$$|\mathrm{Aut}(M^{(i)})| \geqslant |\Sigma| = |\mathrm{Aut}(M)|/2,$$

i.e., the maps $M^{(i)}$ are regular with $\mathrm{Aut}(M^{(0)}) = \mathrm{Aut}(M^{(1)}) \cong \Sigma$.

Now let $\theta \in \mathrm{Aut}(M)$ be any colour-reversing automorphism of M. Then θ' is obviously an automorphism of G satisfying

$$\theta'\pi = \pi\theta. \tag{5.8}$$

Using (5.4) and (5.8) we get

$$\theta' Q^{(i)}(x) = \theta'\pi Q(x_i) = \pi\theta Q(x_i) = \pi Q\theta(x_i) = Q^{(i+1)}\pi\theta(x_i) = Q^{(i+1)}\theta'(x),$$

for every $x \in D(G)$ and $i \in \mathbb{Z}_2$. Hence

$$\theta' Q^{(i)} = Q^{(i+1)}\theta', \tag{5.9}$$

which shows that θ' provides both an isomorphism $M^{(0)} \to M^{(1)}$ and an isomorphism $M^{(1)} \to M^{(0)}$. Recall that besides these isomorphisms we also have $\mathrm{Aut}(M^{(0)}) = \mathrm{Aut}(M^{(1)})$. So Theorem 3.1 now implies that there is an exponent e of G such that

$$Q^{(1)} = (Q^{(0)})^e. \tag{5.10}$$

For $i = 0$ in (5.9) we obtain

$$\theta' Q^{(0)} = Q^{(1)}\theta' = (Q^{(0)})^e\, \theta', \tag{5.11}$$

i.e., e is an exponent of $M^{(0)}$. But (5.9), (5.10) and (5.11) imply that

$$\begin{aligned} Q^{(0)} &= (\theta')^{-1}\, \theta' Q^{(0)} = (\theta')^{-1}\, Q^{(1)}\theta' = (\theta')^{-1}\, (Q^{(0)})^e\, \theta' \\ &= (\theta')^{-1}\, \theta' (Q^{(1)})^e = (Q^{(1)})^e, \end{aligned}$$

whence

$$Q^{(0)} = (Q^{(1)})^e. \tag{5.12}$$

Finally, combining (5.10) and (5.12) we obtain

$$(Q^{(0)})^{e^2} = (Q^{(1)})^e = Q^{(0)},$$

which shows that e is the required involutory exponent of the map $N = M^{(0)}$. Since $(M^{(0)})_e$ clearly coincides with M, the theorem is proved. ∎

To complete our classification of regular embeddings of $\tilde{G}$ in terms of regular embeddings of G it is necessary to know under what conditions two regular embeddings of $\tilde{G}$ are isomorphic. The answer is contained in the following theorem.

THEOREM 5.3. *Let M and N be embeddings of a stable k-valent graph G and let e and f be exponents of G. Then M_e is isomorphic to N_f if and only if M is isomorphic to N and $e \equiv f \pmod{k}$.*

Proof. Let $M = (G, Q)$ and $N = (G, R)$. From the definition of the maps M_e and N_f it follows that

$$\tilde{Q}(x_i) = Q^{e^i}(x)_i \qquad \text{and} \qquad \tilde{R}(x_i) = R^{f^i}(x)_i, \tag{5.13}$$

for any $x_i \in D(\tilde{G})$.

First assume that M is isomorphic to N and $e \equiv f \pmod{k}$. Let $\psi\colon M \to N$ be a map isomorphism. Then $\tilde{\psi}$ is an automorphism of the graph $\tilde{G}$ for which the following equalities hold:

$$\begin{aligned} \tilde{R}\tilde{\psi}(x_i) &= \tilde{R}(\psi(x)_i) = (R^{f^i}\psi(x))_i = (\psi Q^{f^i}(x))_i \\ &= (\psi Q^{e^i}(x))_i = \tilde{\psi}(Q^{e^i}(x)_i) = \tilde{\psi}\tilde{Q}(x_i). \end{aligned}$$

Thus $\tilde{\psi}\colon M_e \to N_f$ is a map isomorphism, proving the sufficiency.

Conversely, let $\psi\colon M_e \to N_f$ be a map isomorphism. We show that then $\psi'\colon M \to N$ is a map isomorphism. To see this, first observe that by (5.13) we have

$$\pi\tilde{R}(x_i) = R^{f^i}\pi(x_i), \tag{5.14}$$

for any $x_i \in D(\tilde{G})$. Now,

$$\psi' Q(x) = \pi\psi(Q(x)_0) = \pi\psi\tilde{Q}(x_0) = \pi\tilde{R}\psi(x_0) = R\pi\psi(x_0) = R\psi'(x).$$

Hence,

$$\psi' Q = R\psi', \tag{5.15}$$

and, consequently, $M \cong N$.

Furthermore, we prove that $\psi'\colon (G, Q^e) \to (G, R^f)$ is a map isomorphism, too. As above, (5.14) implies that

$$\psi' Q^e(x) = \pi\psi(Q^e(x)_1) = \pi\psi\tilde{Q}(x_1) = \pi\tilde{R}\psi(x_1) = R^f\pi\psi(x_1) = R^f\psi'(x).$$

Therefore

$$\psi' Q^e = R^f\psi'. \tag{5.16}$$

If we combine (5.15) and (5.16) we see that

$$\psi' Q^e = R^f\psi' = \psi' Q^f$$

and by multiplying the last equality on the left by $(\psi')^{-1}$ we obtain that $Q^e = Q^f$. Thus $e \equiv f \pmod{k}$, completing the proof. ∎

Theorems 5.2 and 5.3 now imply the following result.

COROLLARY 5.4. *Let G be a connected stable graph. Then the number of non-isomorphic regular embeddings of $\tilde{G}$ is equal to*

$$\sum |\mathrm{Ex}_2(M)|,$$

where M ranges over all non-isomorphic regular embeddings of G.

We conclude this section with a remark concerning the assumption of stability in the statements of our classification theorems (Theorem 5.2 and Theorem 5.3). The reader may have observed that the proofs do not contain any explicit reference to stability except for the identity (4.6) which depends on this assumption (see Theorem 4.2). One might hope that using this identity in the formulations could strengthen the theorems. However, it turns out that at least for simple graphs the condition (4.6) is equivalent to the stability.

THEOREM 5.5. *Let G be a simple graph. Then the following statements are equivalent*:

(1) *G is stable, i.e.,* $\mathrm{Aut}(\tilde{G}) \cong \mathrm{Aut}(G) \times \mathbb{Z}_2$.

(2) $\psi\beta = \beta\psi$ *for every* $\psi \in \mathrm{Aut}(\tilde{G})$.

(3) $\pi\psi(x_0) = \pi\psi(x_1)$, *for every* $\psi \in \mathrm{Aut}(\tilde{G})$ *and every* $x \in D(G)$.

(4) *The group of colour-preserving automorphisms of* $\tilde{G}$ *coincides with* $\Theta(\mathrm{Aut}\, G)$, *where* Θ: $\mathrm{Aut}(G) \to \mathrm{Aut}(\tilde{G})$, $\psi \mapsto \tilde{\psi}$, *is defined by* (4.3).

Proof. The proofs of the implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are contained in the proof of Theorem 4.2.

$(3) \Rightarrow (4)$. Let ψ be a colour-preserving automorphism of $\tilde{G}$. First we prove that if ψ pointwise stabilizes V_0, then $\psi(v_1) = v_1$ for every vertex $v \in V(G)$, i.e., $\psi = \mathrm{id}$. To see this, let v_1 be an arbitrary vertex of V_1 and let $\psi(v_1) = w_1 \in V_1$. Since G has no isolated vertices and is simple, there is an arc uv $(u \neq v)$ terminating at v in G. By the assumption, $\pi\psi(u_0 v_1) = \pi\psi(u_1 v_0)$. Consequently,

$$uw = \pi\psi(u_0 v_1) = \pi\psi(u_1 v_0) = (\pi\psi(u_1)\, v).$$

Thus either $v = u$ or $v = w$. Because $v \neq u$, it follows that $v = w = \pi\psi(v_1)$, and hence $\psi(v_1) = v_1$, as required.

Now, let $\psi \in \mathrm{Aut}(\tilde{G})$ be any colour-preserving automorphism. Then we can define the mapping ψ': $G \to G$ in the same way as in (4.7) and use the proof of Corollary 4.3 to verify that $\psi' \in \mathrm{Aut}(G)$. Clearly, $\psi^{-1}(\psi')^{\sim}$ is a colour-preserving automorphism pointwise stabilizing V_0. As we have proved above, $\psi^{-1}(\psi')^{\sim} = \mathrm{id}$, or equivalently, $\psi = (\psi')^{\sim} \in \Theta(\mathrm{Aut}\, G)$. Thus the subgroup of colour-preserving automorphisms of $\tilde{G}$ coincides with $\Theta(\mathrm{Aut}\, G)$.

$(4) \Rightarrow (1)$. Assume that ψ is a colour-reversing automorphism of $\tilde{G}$. Then $\beta\psi = \xi$ is colour-preserving and $\psi = \beta^2\psi = \beta\xi$, where $\xi \in \Theta(\mathrm{Aut}\, G)$. Since β commutes with every automorphism in $\Theta(\mathrm{Aut}\, G)$ we have $\mathrm{Aut}\, \tilde{G} \cong \Theta(\mathrm{Aut}\, G) \times \langle \beta \rangle$, i.e., G is stable. ∎

6. AUTOMORPHISM AND MONODROMY GROUPS OF THE DERIVED MAPS

Throughout this section, $M = (G, Q)$ denotes a regular embedding of a connected non-bipartite graph G with rotation Q and arc-reversing involution L, and e denotes an involutory exponent of M. An automorphism ψ

of G associated with e gives rise to a group automorphism τ: $\mathrm{Mon}(M) \to \mathrm{Mon}(M)$ defined by setting, for every $w \in \mathrm{Mon}(M)$,

$$\tau(w) = \psi w \psi^{-1}, \tag{6.1}$$

where the right-hand side of (6.1) is to be understood as a product of permutations of $D(G)$. To see that τ is indeed an automorphism of $\mathrm{Mon}(M)$, it suffices to verify that $\psi w \psi^{-1} \in \mathrm{Mon}(M)$, for τ is clearly an isomorphism of $\mathrm{Mon}(M)$ onto a subgroup of the symmetric group on $D(G)$. However, this immediately follows from the fact that

$$\tau(Q) = Q^e \tag{6.2}$$

and

$$\tau(L) = L. \tag{6.3}$$

Actually, if $w \in \mathrm{Mon}(M)$ is expressed in terms of the generators Q and L, then τ simply replaces every occurence of Q in w by Q^e. Since the order of e is 2, equations (6.2) and (6.3) imply that τ is an involution.

These observations can be reversed and generalized to arbitrary exponents and arbitrary maps, see [16]. Here we need only the regular (and involutory) part of the general result.

THEOREM 6.1 [16]. *Let N be a regular map with rotation Q and arc-reversing involution L. Then, an integer f is an exponent of N if and only if there exists an automorphism ζ of* $\mathrm{Mon}(M)$ *such that $\zeta(Q) = Q^f$ and $\zeta(L) = L$. The order of f and that of ζ are equal.*

Now we show that the monodromy group of the derived map M_e is isomorphic to the split extension of $\mathrm{Mon}(M)$ by τ.

THEOREM 6.2. *Let $M = (G, Q)$ be a regular embedding of a connected non-bipartite graph G and let e be an involutory exponent of M. Then* $\mathrm{Mon}(M_e)$ *is isomorphic to the split extension of* $\mathrm{Mon}(M)$ *by the involutory automorphism τ defined in* (6.1).

Proof. Let H be the split extension of $\mathrm{Mon}(M)$ by τ. For the sake of compatibility with the actions of $\mathrm{Mon}(M)$ and $\mathrm{Mon}(M_e)$ we represent the elements of H by pairs (w, λ), where $w \in \mathrm{Mon}(M)$ and $\lambda \in \langle \tau \rangle$ and the multiplication is given by

$$(v, \kappa)(w, \lambda) = (\lambda(v)\, w, \kappa\lambda). \tag{6.4}$$

Consider the elements $(Q, 1)$ and (L, τ) of H. Using the usual notation $\tilde{Q}$ and $\tilde{L}$ for the rotation and the arc-reversing involution of the derived map M_e, we show that $\mathrm{Mon}(M_e) = \langle \tilde{Q}, \tilde{L} \rangle$ is isomorphic to the subgroup $\langle (Q, 1), (L, \tau) \rangle$ of H. We would like to define an isomorphism μ: $\mathrm{Mon}(M) \to \langle (Q, 1), (L, \tau) \rangle$ by setting $\mu(\tilde{Q}) = (Q, 1)$ and $\mu(\tilde{L}) = (L, \tau)$; we also set $\nu(Q, 1) = \tilde{Q}$ and $\nu(L, \tau) = \tilde{L}$. In order to prove that these assignments extend to mutually inverse isomorphisms $\mathrm{Mon}(M_e) \leftrightarrow \langle (Q, 1), (L, \tau) \rangle$ we have to verify two statements:

(i) If $w_1 w_2 \cdots w_q = 1$, with each $w_i \in \{\tilde{Q}, \tilde{L}\}$, then $\mu(w_1) \cdots \mu(w_q) = (1, 1)$.

(ii) If $z_1 z_2 \cdots z_r = (1, 1)$, with each $z_i \in \{(Q, 1), (L, \tau)\}$, then $\nu(z_1) \cdots \nu(z_r) = 1$.

Proof of (i). Let $\tilde{w} = w_1 w_2 \cdots w_q = 1$, where each $w_i \in \{\tilde{Q}, \tilde{L}\}$. Because $\tilde{L}$ is an involution, $\tilde{w}$ can be written in the form

$$\tilde{w} = \tilde{Q}^{c_0} \tilde{L} \tilde{Q}^{c_1} \tilde{L} \cdots \tilde{L} \tilde{Q}^{c_s} = 1. \tag{6.5}$$

Note that when acting on $D(\tilde{G}) = D(G) \times \mathbb{Z}_2$, $\tilde{L}$ changes the second coordinate, while $\tilde{Q}$ does not. Since $\tilde{w}(x_i) = x_i$ for every $x_i \in D(\tilde{G})$, it follows that $\tilde{L}$ occurs in $\tilde{w}$ an even number of times, i.e., s is even. Let $w \in \mathrm{Mon}(M)$ be such that

$$\tilde{w}(x_0) = w(x)_0 = x_0.$$

Then the definitions of $\tilde{Q}$ and $\tilde{L}$ (see (5.1) and (4.2)) imply that

$$w(x) = Q^{c_0} L Q^{ec_1} L Q^{c_2} \cdots L Q^{ec_{s-1}} L Q^{c_s}(x) = x$$

for every $x \in D(G)$. But since $\mathrm{Mon}(M)$ acts regularly on $D(G)$, this shows that

$$w = Q^{c_0} L Q^{ec_1} L Q^{c_2} \cdots L Q^{ec_{s-1}} L Q^{c_s} = 1. \tag{6.6}$$

Now, taking into account that $\tilde{L}$ occurs in $\tilde{w}$ an even number of times, employing (6.6) and multiplying from the right to the left, we find

$$\begin{aligned}
\mu(w_1)\,\mu(w_2) \cdots \mu(w_q) &= (Q, 1)^{c_0}\,(L, \tau)(Q, 1)^{c_1}\,(L, \tau) \cdots (L, \tau)(Q, 1)^{c_s} \\
&= (Q^{c_0}, 1)(L, \tau)(Q^{c_1}, 1)(L, \tau) \cdots (L, \tau)(Q^{c_s}, 1) \\
&= (Q^{c_0} L Q^{ec_1} L \cdots L Q^{ec_{s-1}} L Q^{c_s}, 1) \\
&= (w, 1) = (1, 1).
\end{aligned}$$

This proves (i), i.e., μ: $\mathrm{Mon}(M_e) \to \langle (Q, 1), (L, \tau) \rangle$ is a (surjective) homomorphism.

Proof of (ii). Let $z = z_1 z_2 \cdots z_r = (1, 1)$, where each $z_i \in \{(Q, 1), (L, \tau)\}$. Then z can be written in the form

$$z = (Q, 1)^{d_0} (L, \tau)(Q, 1)^{d_1} (L, \tau) \cdots (L, \tau)(Q, 1)^{d_u} = (1, 1).$$

Since the second coordinate in z is equal to $1 = \tau^0$, it follows that (L, τ) appears in z an even number of times. Performing the products in z from the right to the left we therefore get

$$z = (Q^{d_0} L Q^{ed_1} L \cdots L Q^{d_u}, 1) = (1, 1). \tag{6.7}$$

Set

$$z^{(0)} = Q^{d_0} L Q^{ed_1} L \cdots L Q^{ed_{u-1}} L Q^{d_u},$$
$$z^{(1)} = Q^{ed_0} L Q^{d_1} L \cdots L Q^{d_{u-1}} L Q^{ed_u} = \tau(z^{(0)})$$

and

$$\tilde{z} = \nu(z_1)\, \nu(z_2) \cdots \nu(z_r) = \tilde{Q}^{d_0} \tilde{L} \tilde{Q}^{d_1} \tilde{L} \cdots \tilde{L} \tilde{Q}^{d_{u-1}} \tilde{L} \tilde{Q}^{d_u}.$$

Then (6.7) implies that $z^{(0)} = 1$ and hence also $z^{(1)} = \tau(z^{(0)}) = 1$. Since $\tilde{z}$ has an even number of occurrences of $\tilde{L}$, it preserves the second coordinate of each element of $D(\tilde{G}) = D(G) \times \mathbb{Z}_2$. Therefore for every $x \in D(G)$ we obtain:

$$\tilde{z}(x_0) = (z^{(0)}(x))_0 = x_0$$
$$\tilde{z}(x_1) = (z^{(1)}(x))_1 = x_1$$

Realizing that $\mathrm{Mon}(M_e)$ acts regularly on $D(\tilde{G})$ we conclude that $\tilde{z} = 1$. Thus $\nu\colon \langle (Q, 1), (L, \tau) \rangle \to \mathrm{Mon}(M_e)$ is a homomorphism inverse to μ, proving (ii).

So far we have shown that $\mathrm{Mon}(M_e)$ is isomorphic to the subgroup $\langle (Q, 1), (L, \tau) \rangle$ of H. But

$$|\mathrm{Mon}(M_e)| = |\langle (Q, 1), (L, \tau) \rangle| \leqslant |H| = 2\, |\mathrm{Mon}\, M| = |\mathrm{Mon}(M_e)|,$$

which implies that $\langle (Q, 1), (L, \tau) \rangle = H$, completing the proof of the theorem. ∎

Recall that the monodromy group of every regular map is isomorphic to its automorphism group. Thus the above theorem gives also a description of the automorphism group of the derived map. In particular, it implies the following:

COROLLARY 6.3. *The automorphism group* $\mathrm{Aut}(M_e)$ *is isomorphic to a split extension of* $\mathrm{Aut}(M)$ *by* $\mathbb{Z}_2$.

As to when this extension is just the direct product, we have

THEOREM 6.4. *Under the assumptions of Theorem* 6.2, *the following statements are equivalent*:

(1) $\mathrm{Aut}(M_e) \cong \mathrm{Aut}(M) \times \mathbb{Z}_2$.

(2) *The natural mapping* $\pi: M_e \to M$ *is a double covering projection of* M.

(3) $e \equiv 1$ *modulo the valency of the underlying graph of* M.

Proof. The equivalence $(1) \Leftrightarrow (3)$ is an easy consequence of Theorem 6.2. According to (5.1), the rotation $\tilde{Q}$ of M_e is a lifting (in the usual topological sense) of the rotation Q of M if and only if (3) is fulfilled. This establishes the equivalence $(2) \Leftrightarrow (3)$. ∎

Finally, we would like to clarify which extensions of the monodromy group of the base map by an involutory automorphism occur as the monodromy groups of the derived maps. Obviously, (6.2) and (6.3) provide certain necessary conditions. In the next theorem we show that these conditions are also sufficient. More precisely, we have

THEOREM 6.5. *Let* M *be a regular embedding of a connected non-bipartite graph* G *with rotation* Q *and arc-reversing involution* L. *Let* ζ *be an involutory automorphism of* $\mathrm{Mon}(M)$ *such that* $\zeta(L) = L$ *and* $\zeta(Q) = Q^e$ *for some integer* e. *Then there is a regular embedding* $\tilde{M}$ *of* $\tilde{G}$ *such that* $\mathrm{Mon}(\tilde{M})$ *is isomorphic to the split extension of* $\mathrm{Mon}(M)$ *by* ζ.

Proof. Theorem 6.1 implies that e is an involutory exponent of M. So we can construct the derived embedding M_e of $\tilde{G}$. By Theorem 6.2, there is an involutory automorphism τ of $\mathrm{Mon}(M)$ such that $\mathrm{Mon}(M_e)$ is a split extension of $\mathrm{Mon}(M)$ by τ. Taking into account that by (6.2) and (6.3), $\tau(Q) = Q^e = \zeta(Q)$ and $\tau(L) = L = \zeta(L)$, we deduce that $\zeta = \tau$, and the proof is complete. ∎

7. EXPONENT GROUPS OF THE DERIVED MAPS AND THE PETRIE DUALITY

The primary aim of this section is to prove the following

THEOREM 7.1. *Let* G *be a connected non-bipartite stable graph and let* M *be an embedding of* G. *Then the map* M_e, *where* e *is an exponent of* G, *has the same exponents as* M. *Consequently*,

$$\mathrm{Ex}(M_e) = \mathrm{Ex}(M).$$

Proof. Since the graphs G and $\tilde{G}$ have the same valency, it is sufficient to prove that an integer f is an exponent of M if and only if f is an exponent of M_e.

Let f be an arbitrary exponent of M and let ψ be an automorphism of the graph G associated with f. Then

$$\psi Q = Q^f \psi.$$

Now, for every $x_i \in D(\tilde{G})$ we have

$$\tilde{\psi}\tilde{Q}(x_i) = \tilde{\psi}(Q^{e^i}(x)_i) = (\psi Q^{e^i}(x))_i = ((Q^{e^i})^f\, \psi(x))_i = \tilde{Q}^f(\psi(x)_i) = \tilde{Q}^f\tilde{\psi}(x_i).$$

Thus f is an exponent of M_e and $\tilde{\psi}$ is an automorphism of $\tilde{G}$ associated with f.

For the converse, assume that f is an exponent of M_e and $\psi \in \mathrm{Aut}(\tilde{G})$ is an automorphism associated with f; i.e., $\psi\tilde{Q} = \tilde{Q}^f\psi$. Realizing that $\psi' \in \mathrm{Aut}(G)$ and that $(M^{(0)})_e = M$, from (5.6), (5.7) and (5.15) we get

$$\psi' Q(x) = \psi' \pi \tilde{Q}(x_0) = \pi\psi\tilde{Q}(x_0) = \pi\tilde{Q}^f(\psi'(x)_0) = Q^f\pi(\psi'(x)_0) = Q^f\psi'(x).$$

Therefore f is an exponent of M and $\psi' \in \mathrm{Aut}(G)$ is associated with f. This completes the proof. ∎

COROLLARY 7.2. *Let M be a regular map with stable underlying graph. Then M_e is reflexible if and only if M is reflexible.*

From each reflexible regular map one can form another regular map with the same underlying graph by replacing its faces with Petrie polygons. A *Petrie polygon* on an arbitrary map M is a "zig-zag" closed walk where any two but not three consecutive arcs share a face. Each edge is contained in two (not necessarily distinct) Petrie polygons, so the collection of all Petrie polygons of M gives rise in the obvious way to a new map $\mathbf{p}(M)$. It is usually called the *Petrie dual* of M, the term being justified by the fact that $\mathbf{pp}(M) \cong M$. (For details concerning this and other operations on maps we refer the reader to Jones and Thornton [11].)

If M is regular, then $\mathbf{p}(M)$ is regular only if M is reflexible—even in the case when both M and $\mathbf{p}(M)$ are orientable. If M is orientable, then $\mathbf{p}(M)$ is orientable precisely when the underlying graph of M is bipartite. In this case, $\mathbf{p}(M)$ is obtained by simply switching all the local rotations in one of the partite sets.

Now consider a regular map M_e derived from a regular map M by using an involutory exponent $e \in \mathrm{Ex}_2(M)$. Since M_e is bipartite, $\mathbf{p}(M)$ is orientable and, by the above switching argument, coincides with M_{-e}. Assume that M_e is reflexible (so that M_{-e} is regular). Since, by Theorem 7.1, M is reflexible as well, both e and -1 are involutory exponents of M. Thus

$-e = -1.e$ is an involutory exponent of M, and the regularity of M_{-e} is also confirmed by Theorem 5.1. Moreover, Theorem 5.3 implies that M_e is self-Petrie if and only if $e \equiv -e$ modulo the valency of M. Since e is involutory, this is only possible when M is 2-valent, i.e., M is an embedding of a cycle in the 2-sphere.

To close this section we revisit the example from Introduction. Let M be the tetrahedron, i.e., the triangular embedding of K_4 in the 2-sphere. Since M is reflexible, Theorem 5.1 yields (at least) two regular embeddings for the canonical double covering, namely M_1 and M_{-1}, the hexagonal and the quadrilateral embedding of the 3-cube in the torus and the 2-sphere, respectively. As follows from our discussion above, these maps are the Petrie duals of each other.

8. Regular Embeddings of $K_n \otimes K_2$

In the next two sections we employ our classification theorems to extend the list of graphs for which classification of all regular embeddings is known. Surprisingly enough, complete graphs K_n seem to provide the only non-trivial infinite family of graphs for which all the regular maps are known [1, 7, 8, 9, 21]. Using this classification and our Theorems 5.1–5.3 we determine in this section all regular embeddings of the tensor product $K_n \otimes K_2$ which is sometimes called the *cocktail-party graph*.

The regular embeddings of the complete graph K_n are most conveniently described as embeddings of Cayley graphs $C(F, F^*)$ based on the additive groups of the finite fields $F = GF(p^l)$ with the generator set $F^* = F - \{0\}$; i.e., the vertices of K_n are labelled with the elements of F, and two vertices a and b are adjacent if $a - b \in F^*$.

For any primitive element t of $GF(p^l)$, let L_t be the embedding of $K_n = C(F, F^*)$ given by the rotation Q whose action on an arc (a, b), $a, b \in GF(p^l)$, $a \neq b$, is

$$Q(a, b) = (a, a + (b - a)\, t).$$

It has been shown that this embedding is regular [1, 9] and that each regular embedding of K_n (on an orientable surface) is isomorphic to some L_t [9]. In particular, K_n admits a regular embedding only when n is a prime power. Moreover, two maps L_s and L_t are isomorphic if and only if the primitive elements s and t of $F = GF(p^l)$ are conjugate under the Galois group $\mathrm{Gal}(F)$ of F (over its prime field $GF(p)$), see [9].

In order to be able to give a classification of the regular embeddings of $K_n \otimes K_2$ we first observe that K_n is stable for every $n \geqslant 3$.

PROPOSITION 8.1. *The complete graph K_n is stable for every $n \geqslant 3$.*

Proof. It is easy to see that $K_n \otimes K_2$ is isomorphic to the complete bipartite graph $K_{n,n}$ minus a 1-factor. Since $n \geqslant 3$, it is connected, and hence every colour-preserving automorphism of $K_n \otimes K_2$ is uniquely determined by a permutation of the missing 1-factor. Thus the group of all colour-preserving automorphisms of $K_n \otimes K_2$ is isomorphic to the symmetric group S_n, implying that $\mathrm{Aut}(K_n \otimes K_2) \cong \mathrm{Aut}(K_n) \times \mathbb{Z}_2$. ▮

By Theorem 5.2, the classification of regular embeddings of $K_n \otimes K_2$ now reduces to the calculation of the groups of involutory exponents of the maps L_t. As we show in [16], the exponent group of the map L_t is isomorphic to the Galois group $\mathrm{Gal}(F)$. Since $\mathrm{Gal}(GF(p^l))$ is a cyclic group of order l generated by the Frobenius automorphism

$$\varrho\colon z \mapsto z^p, \qquad z \in F,$$

there is a natural isomorphism $\mathrm{Gal}(F) \to \mathrm{Ex}(L_t)$ given by

$$\Phi\colon \varrho^i \mapsto [p^i], \qquad 0 \leqslant i \leqslant l-1, \tag{8.1}$$

where $[m]$ denotes the residue class of an integer $m \pmod{p^l - 1}$. Thus $\mathrm{Ex}(L_t)$ has a non-trivial involution if and only if l is even; the involution in question is obviously $\Phi(\varrho^{l/2})$, and the corresponding exponent is $p^{l/2}$. To summarize, in [16] we prove the following:

THEOREM 8.2 [16]. *An integer e is an exponent of the embedding L_t of K_n, $n = p^l$, if and only if $e \equiv p^i \pmod{p^l - 1}$, $i = 0, 1, ..., l-1$. Consequently, $\mathrm{Ex}(L_t)$ is equal to the subgroup $\{[p^i];\ 0 \leqslant i \leqslant l-1\}$ of the multiplicative group $\mathbb{Z}^*_{p^l-1}$, which in turn is isomorphic to $\mathrm{Gal}(GF(p^l))$.*

COROLLARY 8.3 [16]. *If l is even, then $\mathrm{Ex}_2(L_t)$ is equal to the subgroup $\{1, p^{l/2}\}$ of $\mathbb{Z}^*_{p^l-1}$. If l is odd, then $\mathrm{Ex}_2(L_t)$ is trivial.*

The results of the previous section imply that every regular embedding of $K_n \otimes K_2$ is of the form $(L_t)_e$, where $e = 1$ or $e = p^{l/2}$ (whenever $l/2$ is an integer). Now we want to give a more explicit description of the resulting maps.

We shall continue with the notation introduced above: $n = p^l$ is a prime power, $n \geqslant 3$, $F = GF(p^l)$ and $K_n = C(F, F^*)$. Take a primitive element t of F and an involution $\theta \in \mathrm{Gal}(F)$. Define the rotation Q for $\tilde{K}_n = K_n \otimes K_2$ by setting

$$Q(a_i, b_{i+1}) = (a_i, (a + (b-a)\,\theta^i(t))_{i+1}), \tag{8.2}$$

and denote the constructed map $(K_n \otimes K_2, Q)$ by $M(t, \theta)$. We show that the embedding $M(t, \theta)$ of $K_n \otimes K_2$ is regular. In view of Theorem 5.1, it suffices to prove that $M(t, \theta) = (L_t)_e$ for an involutory exponent e of the map L_t. By (8.2) we have

$$Q^{(0)}(a, b) = \pi Q(a_0, b_1) = \pi(a_0, (a + (b-a)\, t)_1) = (a, a + (b-a)\, t).$$

Thus the rotation $Q^{(0)}$ coincides with the rotation of the map L_t. Moreover,

$$Q^{(1)}(a, b) = \pi Q(a_1, b_0) = \pi(a_1, (a + (b-a)\, \theta(t))_0) = (a, a + (b-a)\, \theta(t)).$$

Since θ is an involution in $\mathrm{Gal}(F)$ and $\Phi\colon \mathrm{Gal}(F) \to \mathrm{Ex}(L_t)$, $\varrho^i \mapsto [p^i]$, $0 \leqslant i \leqslant l-1$, is an isomorphism, every representative e of the residue class $\Phi(\theta) \in \mathrm{Ex}(L_t)$ is an involutory exponent. Therefore

$$Q^{(1)}(a, b) = (a, a + (b-a)\, \theta(t)) = (a, a + (b-a)\, t^e) = (Q^{(0)})^e\, (a, b),$$

and hence

$$M(t, \theta) = (L_t)_e.$$

So $M(t, \theta)$ is a regular map.

Now we are prepared to state and prove the theorem giving the classification of regular embeddings of $K_n \otimes K_2$.

THEOREM 8.4. *Every regular embedding of the graph $K_n \otimes K_2, n \geqslant 3$, is isomorphic to one of the maps $M(t, \theta)$, where t is a primitive element of $GF(n)$ and $\theta \in \mathrm{Gal}(GF(n))$ is an involution. In particular, $K_n \otimes K_2$ admits a regular embedding if and only if n is a prime power. Moreover, two such maps $M(s, \sigma)$ and $M(t, \theta)$ are isomorphic if and only if $\sigma = \theta$ and s is conjugate with t under* $\mathrm{Gal}(GF(n))$.

Proof. Let N be a regular embedding of $K_n \otimes K_2$. Since K_n is stable, by Theorem 5.2 there exists a regular embedding M of K_n and an involutory element e of M such that $N \cong M_e$. By [9], the embedding M of K_n can be expressed as an embedding L_t of the Cayley graph $C(F, F^*)$, where $F = GF(n)$, $n = p^l$ is a prime power and t is a primitive element of F, with rotation

$$Q(a, b) = (a, a + (b-a)\, t),$$

$a, b \in F$, $a \neq b$. Note that

$$Q^j(a, b) = (a, a + (b-a)\, t^j),$$

for any integer j.

Denoting the arc (a_i, b_{i+1}) in $K_n \otimes K_2 \cong \tilde{K}_n$ by $(a, b)_i$, the definition of the rotation $\tilde{Q}$ of M_e implies that

$$\tilde{Q}(a_0, b_1) = (Q(a, b))_0 = (a, a + (b-a)\, t)_0 = (a_0, (a + (b-a)\, t)_1)$$

and

$$\tilde{Q}(a_1, b_0) = (Q^e(a, b))_1 = (a, (a + (b-a)\, t^e)_1 = (a_1, (a + (b-a)\, t^e)_0).$$

Hence

$$\tilde{Q}(a_i, b_{i+1}) = (a_i, (a + (b-a)\, t^{e^i})_{i+1}), \quad i = 0, 1.$$

Let $[e] \in \mathrm{Ex}(M)$ be the residue class coressponding to the exponent e of M. Recall that Φ: $\mathrm{Gal}(F) \to \mathrm{Ex}(M)$ is the isomorphism described in (8.1), then $\theta = \Phi^{-1}([e]) \in \mathrm{Gal}(F)$ is an involutory automorphism of F such that $t^{e^i} = \theta^i(t)$. Therefore $N \cong M(t, \theta)$.

Let $M(s, \sigma)$ and $M(t, \theta)$ be isomorphic maps. Then the considerations before this theorem imply that $M(s, \sigma) = (L_s)_e$, where e represents the residue class of $\Phi(\sigma) \in \mathrm{Ex}_2(M_s)$. Similarly, $M(t, \theta) = (L_t)_f$, where f represents $\Phi(\theta)$. According to Theorem 5.3, $L_s \cong L_t$ and $e \equiv f \pmod{p^l - 1}$. By [9], $L_t \cong L_s$ if and only if s and t are conjugate under $\mathrm{Gal}(F)$. ▮

As regards the number of non-isomorphic regular embeddings of $\tilde{K}_n \cong K_n \otimes K_2$, we have the following result.

THEOREM 8.5. *The number of non-isomorphic regular embeddings of* $K_n \otimes K_2$, $n = p^l \geqslant 3$, *p a prime, is equal to*

$$\begin{array}{ll} 2\phi(p^l - 1)/l, & \textit{if } l \textit{ is even}, \\ \phi(p^l - 1)/l, & \textit{if } l \textit{ is odd}, \end{array}$$

$\phi(x)$ *being Euler's function of an integer* x.

Proof. Let h be the number of regular embeddings of $K_n \otimes K_2$. Corollary 5.4 implies that

$$h = \sum |\mathrm{Ex}_2(M)|,$$

where M ranges over all non-isomorphic regular embeddings of K_n. By Corollary 8.3, for any regular embedding M of K_n we have $|\mathrm{Ex}_2(M)| = 2$ or $|\mathrm{Ex}_2(M)| = 1$ according as l is even or odd. Thus $h = 2h'$ or $h = h'$, where h' is the number of non-isomorphic regular embeddings of K_n. But James and Jones show in [9] that $h' = \phi(p^l - 1)/l$, so the theorem follows. ▮

COROLLARY 8.6. *The 3-cube graph Q_3 underlies exactly two (orientable) regular maps, one of them being the unique quadrilateral embedding in the 2-sphere and the other being the hexagonal embedding in the torus, the Petrie dual of the former map.*

Proof. It is sufficient to realize that $Q_3 = K_4 \otimes K_2$. ▮

As we have seen, every regular embedding of $K_n \otimes K_2$ is one of the maps $M(t, \theta)$, where t is a primitive element of $GF(n)$, $n = p^l$, and θ is an involution in $\mathrm{Gal}(F)$. From this point of view, the regular embeddings $M(t, \theta)$ of $K_n \otimes K_2$ fall into two natural classes: those with θ trivial and those with θ non-trivial. Each map $M(t, \mathrm{id})$ of the first type is easily seen to be a double covering over the regular map L_t with underlying graph K_n. Maps of the first type exist for each n being a prime power. In contrast to this, no map $M(t, \varrho^{l/2})$ of the second type covers an orientable regular embedding of K_n. Thus maps of the second type apparently provide a new infinite class of regular maps. These maps exist only if n is an even power of a prime. It is quite surprising that for $K_4 \otimes K_2$ the "non-trivial" map of the second type is the well known quadrilateral embedding in the sphere while the "trivial" map of the first type is the less popular hexagonal embedding in the torus.

Theorem 5.5 and Proposition 8.1 show that

$$\mathrm{Ex}(M(t, \theta)) = \mathrm{Ex}(L_t) = \{1, p, p^2, ..., p^{l-1}\} \subseteq \mathbb{Z}^*_{p^l - 1}. \tag{8.3}$$

In particular, $M(t, \theta)$ is reflexible if and only if $n \leqslant 4$. This follows directly from (8.3) or via Corollary 7.2 from the results of James and Jones [9] about regular embeddings of K_n.

We conclude the discussion of the regular embeddings of $K_n \otimes K_2$ by giving formulas to compute the face length of $M(t, \theta)$. We distiguish two cases according to whether θ is trivial or not. James and Jones [9] showed that the embedding L_t of K_n has face length either $n-1$ or $(n-1)/2$, according as $n \not\equiv 3 \pmod 4$ or $n \equiv 3 \pmod 4$. Using the fact that $M(t, \mathrm{id})$ is a double covering over L_t, it can readily be computed that the face length of $M(t, \mathrm{id})$ is either $n-1$ if n is odd or $2(n-1)$ if n is even.

The case $M = M(t, \varrho^{l/2})$ is much more interesting. Let $s = \varrho^{l/2}(t) = t^{p^{l/2}}$. By the definition of the rotation of M, the initial portion of the face boundary containing the arc $(0_0, 1_1)$ is

$$(0_0, 1_1), (1_1, (1-s)_0), ((1-s)_0, (1-s+st)_1),$$

$$((1-s+st)_1, (1-s+st-s^2t)_0),$$

$$((1-s+st-s^2t)_0, (1-s+st-s^2t+s^2t^2)_1), \ldots.$$

Obviously, the face length d of M is even. Thus, if the face length is $d=2k$, then

$$0=1-s+st-s^2t+s^2t^2-\cdots+s^{k-1}t^{k-1}-s^kt^{k-1}$$
$$=(1-s)(1+st+(st)^2+\cdots+(st)^{k-1})=(1-s)\,\frac{(st)^k-1}{st-1}, \qquad \text{or } st=1.$$

Since $s=t^{p^{l/2}}=t^{\sqrt{n}}$ we have

$$0=(1-t^{\sqrt{n}})\,\frac{t^{k(\sqrt{n}+1)}-1}{t^{\sqrt{n}+1}-1},$$

or

$$t^{\sqrt{n}+1}=1.$$

The second case occurs if and only if $n=4$. Direct computations yield that $d=4$. Otherwise, $t^{k(\sqrt{n}+1)}=1$, whence

$$k=\frac{\operatorname{lcm}(\sqrt{n}+1,n-1)}{\sqrt{n}+1}=\frac{n-1}{\gcd(\sqrt{n}+1,n-1)}.$$

Consequently, the face length of $M(t,\varrho^{l/2})$ is $d=2(n-1)/\gcd(\sqrt{n}+1,n-1)$ for $n>4$, and $d=4$ for $n=4$.

9. Regular Embeddings of Dipoles

The *n-dipole* D_n is a graph which consists of two vertices joined by n parallel edges. If such a graph is cellularly embedded in some closed surface, then the dual map has two faces. Regular maps having two faces were first considered by Brahana [2], later discussed by Coxeter and Moser in their well-known book [3, pp. 113–115], Garbe [5], and by Vince [18]. Using a purely group-theoretical language, Garbe [5, Satz 6.2] found a necessary and sufficient condition for the existence of a two-face regular map on an orientable surface of given genus. Since the dual of a regular map is again a regular map, these authors actually dealt with regular embeddings of dipoles.

Let u_0 and u_1 be the vertices of D_n. Every edge x of D_n gives rise to two mutually reverse arcs originating at u_0 and u_1, respectively. If R is any cyclic permutation of edges of D_n, and e is an integer relatively prime to n, then the permutation

$$Q(x_i)=(R^{e^i}(x))_i \tag{9.1}$$

of the arcs of D_n is clearly a rotation for D_n. Denoting the resulting map by $M(n, e)$, we can now state a result which gives the complete classification of regular embeddings of dipoles, and thereby that of regular maps with two faces.

THEOREM 9.1. *An embedding of the n-dipole is regular if and only if it is isomorphic to the map $M(n, e)$ for some integer e such that $e^2 \equiv 1 \pmod{n}$. Moreover, $M(n, e) \cong M(n, f)$ if and only if $e \equiv f \pmod{n}$.*

Proof. To see that this theorem is an easy consequence of our classification results proved in Section 5 we only need to make several simple remarks.

Firstly, if the *n-semistar* Ss_n is a graph consisting of a single vertex u and n semiedges incident with u, then D_n is clearly isomorphic to the canonical double covering of Ss_n, whereby a semiedge x of Ss_n lifts to the arcs x_0 and x_1 of D_n. Trivially, $\mathrm{Aut}(D_n) \cong S_n \times \mathbb{Z}_2 \cong \mathrm{Aut}(Ss_n) \times \mathbb{Z}_2$, which means that the n-semistar is a stable graph. (For n even, D_n is also the canonical double covering of the bouquet of $n/2$ circles, however, this graph is unstable.)

Secondly, any cyclic permutation R of the semiedges of Ss_n determines a regular map (which is even reflexible), and any two (regular) embeddings of Ss_n are isomorphic.

Thirdly, if $M(n) = (Ss_n, R)$ is a regular embedding of Ss_n, then an integer e is an exponent of $M(n)$ precisely when R^e is a cyclic permutation, i.e., when $\gcd(n, e) = 1$. (This implies that $\mathrm{Ex}(M(n)) = \mathbb{Z}_n^* = \mathrm{Ex}(Ss_n)$.) In particular, e is an involutory exponent of $M(n)$ if and only if $e^2 \equiv 1 \pmod{n}$.

Finally, note that the rotation Q given by (9.1) coincides with the rotation $\tilde{R}$ derived from R according to (5.1). Thus the map $M(n, e)$ is nothing but the map $(M(n))_e$, and Theorem 9.1 follows. ∎

Corollary 5.4 implies that the number of non-isomorphic regular embeddings of D_n is equal to the number of solutions of the congruence $e^2 \equiv 1 \pmod{n}$. This number is well known, see, e.g., [14, p. 84]. So we get the following formulas for the number of regular embeddings of D_n.

COROLLARY 9.2. *Let $n = 2^{\alpha} p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ ($\alpha_i > 0$, for $i = 1, ..., k$) be the canonical decomposition of the integer n. If m is the number of non-isomorphic regular embeddings of D_n, then*

$$m = 2^k, \qquad \text{if} \quad \alpha \leqslant 1$$

$$m = 2^{k+1}, \qquad \text{if} \quad \alpha = 2$$

$$m = 2^{k+2}, \qquad \text{if} \quad \alpha \geqslant 3.$$

By Theorem 7.1, $\mathrm{Ex}(M(n, e)) = \mathrm{Ex}(Ss_n, R) = \mathbb{Z}_n^*$, in particular every regular embedding of D_n is reflexible. In order to compute the face length of the map $M(n, e)$ consider the permutation $Q\tilde{L}$, $\tilde{L}(x, i) = (x, i+1)$, $x \in D(Ss_n)$ and $i \in \mathbb{Z}_2$, cyclically permuting the arcs on the face boundaries. It is easy to see that $(Q\tilde{L})^2\,(x, i) = (R^{e+1}(x), i)$. Hence the face length d of the map $M(n, e)$ is equal to twice the order of R^{e+1}. Thus

$$d = \frac{2.\mathrm{lcm}(e+1, n)}{e+1} = \frac{2n}{\gcd(n, e+1)}.$$

Theorem 9.1 has an interesting application to the generalized Petersen graphs, whose details are given in [15]. The generalized Petersen graph $GP(n, k)$, $n \geqslant 3$, $1 \leqslant k < n/2$, is a cubic graph with vertex-set $\{u_i;\ i \in \mathbb{Z}_n\} \cup \{v_i;\ i \in \mathbb{Z}_n\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k};\ i \in \mathbb{Z}_n\}$. If e is relatively prime to n, then $GP(n, k)$ can easily be obtained from the map $M(n, e)$ with underlying graph D_n by expanding every vertex u_i, $i = 0, 1$, to a cycle $C(u_i)$ on the supporting surface in such a way that the cyclic order of the edges originally incident with u_i remains unchanged. A similar operation can be applied to an arbitrary map M to obtain a cubic graph $T(M)$. The graph $T(M)$ can be endowed with a natural structure of a Schreier coset graph, which becomes a Cayley graph if and only if M is regular. In particular, the sufficient condition in Theorem 9.1 implies that $GP(n, e)$ is a Cayley graph whenever $e^2 \equiv 1 \pmod{n}$. A more sophisticated argument based on Theorem 9.1 yields the converse statement, implying that $GP(n, k)$ is a Cayley graph if and only if $k^2 \equiv 1 \pmod{n}$. Frucht, Graver and Watkins [4] showed that $GP(n, k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$ or $(n, k) = (10, 2)$. It follows that the generalized Petersen graphs $GP(n, k)$ with $k^2 \equiv -1 \pmod{n}$ provide a simple and interesting infinite family of cubic vertex-transitive graphs that are not Cayley graphs. It is not necessary to emphasize that the methods of Frucht, Graver and Watkins [4] are different from those employed in [15] and here.

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